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## Some results on reflection principles in fragments of Peano arithmetic

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This is an abstract of the paper [3]. Let  $IS_k$  denote the fragment of Peano arithmetic, whose axioms are Peano's axioms with induction restricted to  $\Sigma_k$ -formulas. We will develop a proof-theoretic study of various principles of fragments of Peano arithmetic such as reflection principles, transfinite inductions, well-ordering principles and large set principles, and compare their proof-theoretic strength.

Let  $Pr_k(x)$  be a canonical representation of the provability predicate for  $IS_k$ . Let  $IS_k + T_m$  be the arithmetic obtained from  $IS_k$  by adding all true  $\Pi_m$ -sentences as additional axioms. In the following, the function symbol  $\beta$  represents the Gödel's  $\beta$ -function. So,  $\beta(x, i) = y$  means that  $y$  is the  $i$ -th element of the sequence coded by  $x$ . We will define ordinals  $\omega_n$  by  $\omega_0 = 1$  and  $\omega_{n+1} = \omega^{\omega_n}$ . For each positive integer  $n$ , let  $<_n$  be a canonical primitive recursive well-ordering of natural numbers of order-type  $\omega_n$ . Sometimes, we will omit the subscript  $n$  of  $<_n$ , when no confusions will occur. For each natural number  $x$ , let  $|x|_n$  denote the ordinal  $\alpha$  represented by  $x$  in the well-ordering  $<_n$ . By abuse of notations, we will often write the

ordinal  $\alpha$  in place of  $x$ , when  $\alpha = |x|_n$ .

We will consider the following seven principles:

1)  $RFN_{\Sigma_m}(I\Sigma_k)$  (  $\Sigma_m$ -uniform reflection principle of  $I\Sigma_k$  ):

For any  $\Sigma_m$ -formula  $\varphi(x)$ ,  $\forall x \text{Pr}_k([\varphi(\dot{x})]) \supset \forall x \varphi(x)$ .

2)  $\text{Con}(I\Sigma_k + T_m)$  ( consistency of  $I\Sigma_k + T_m$  ).

3)  $TI_{\Pi_m}[\omega_n]$  ( transfinite induction up to  $\omega_n$  for  $\Pi_m$ -formulas ):

For any  $\Pi_m$ -formula  $\psi(x)$ ,

$\forall \alpha < \omega_n [ \forall \gamma ( \forall \beta ( \beta <_n \gamma \supset \psi(\beta) ) \supset \psi(\gamma) ) \supset \psi(\alpha) ]$ .

4)  $WOP_{\Sigma_m}[\omega_n]$  ( well-ordering principle of  $\omega_n$  for  $\Sigma_m$ -formulas ):

Let  $\theta$  be a formula containing at least two free variables and let  $F(\theta)$  denote the formula  $\forall x \exists ! y \theta(x, y)$ . Then,  $WOP_{\Sigma_m}[\omega_n]$  is the schema: for any  $\Sigma_m$ -formula  $\theta$  containing at least two free variables,

$F(\theta) \supset \exists x \exists y \exists z ( \theta(x, y) \wedge \theta(x+1, z) \wedge \neg ( z <_n y ) )$ .

Roughly speaking,  $WOP_{\Sigma_m}[\omega_n]$  means that if a function  $f : \mathbb{N} \rightarrow \omega_n$  is represented by some  $\Sigma_m$ -formula then the sequence  $f(0), f(1), f(2), \dots$  is not strictly descending with respect to  $<_n$ .

5)  $LSP_{\Sigma_m}[\omega_n]$  (  $\omega_n$ -large set principle for  $\Sigma_m$ -formulas ).

Let  $[x, y]$  denote the set  $\{ z : x \leq z \leq y \}$  of natural numbers. Suppose that  $\theta$  is a formula containing at least two free variables. Then,  $SIF(\theta)$  is the formula

$\forall x \exists ! y \theta(x, y) \wedge \forall x \forall y \forall z ( ( \theta(x, y) \wedge \theta(x+1, z) ) \supset y < z )$ ,

which means that  $\theta$  is the graph of a strictly increasing function. Let  $\{ \gamma \}(x)$  be the fundamental sequence defined in [1]. Let  $fs_n$  be the function symbol which represents the

primitive recursive function  $fs_n$  such that  $fs_n(u, x) = w$  if and only if  $\{|u|_n\}(x) = |w|_n$ . Now, for each  $\alpha < \omega_n$  and each formula  $\theta$ , we will abbreviate the following formula

$$\exists z [ \beta(z, 0) = \alpha \wedge \forall w < (y-x) \exists u \exists t ( \beta(z, w) = u \wedge \theta(x+w, t) \wedge \beta(z, w+1) = fs_n(u, t) ) \wedge \beta(z, y-x) = 0 ],$$

to ' $[x, y]$  is  $(\alpha, \theta)$ -large'. Then,  $LSP_{\Sigma_m}[\omega_n]$  is the schema; for any  $\Sigma_m$ -formula  $\theta$  containing at least two free variables,

$$SIF(\theta) \supset \forall \alpha < \omega_n \forall x \exists y ( [x, y] \text{ is } (\alpha, \theta)\text{-large} ).$$

Clearly,  $LSP_{\Sigma_m}[\omega_n]$  means that if a function  $f$  is represented by some  $\Sigma_m$ -formula then for any  $\alpha < \omega_n$   $\forall x \exists y ( [x, y] \text{ is } (\alpha, f)\text{-large} )$  holds. Here, we say that  $[x, y]$  is  $(\alpha, f)$ -large if the set  $f([x, y])$  is  $\alpha$ -large ( see [2] ).

6)  $WOP^*_{\Sigma_m}[\omega_n]$  ( well-ordering principle of  $\omega_n$  for  $\Sigma_m$ -definable functions ): For any  $\Sigma_m$ -formula  $\theta$  containing at least two free variables,

$$Pr_m([F(\theta)]) \supset \exists x \exists y \exists z ( \theta(x, y) \wedge \theta(x+1, z) \wedge \neg ( z <_n y ) ).$$

7)  $LSP^*_{\Sigma_m}[\omega_n]$  (  $\omega_n$ -large set principle for  $\Sigma_m$ -definable functions ): For any  $\Sigma_m$ -formula  $\theta$  containing at least two free variables,

$$Pr_m([SIF(\theta)]) \supset \forall \alpha < \omega_n \forall x \exists y ( [x, y] \text{ is } (\alpha, \theta)\text{-large} ).$$

Then, we have the following theorems.

THEOREM 1. Let  $m$  be positive integer.

$$1) \quad I\Sigma_1 + RFN_{\Sigma_{m+1}}(I\Sigma_{m+n-1}) \vdash TI_{\Pi_m}[\omega_{n+1}] \quad \text{for } n > 0.$$

2) The following three theories are equivalent ( $n \geq 0$ ):

a.  $I\Sigma_1 + TI_{\Pi_m}[\omega_{n+1}]$ .

b.  $I\Sigma_m + WOP_{\Sigma_m}[\omega_{n+1}]$ .

c.  $I\Sigma_m + LSP_{\Sigma_m}[\omega_{n+1}]$ .

3)  $I\Sigma_1 + TI_{\Pi_m}[\omega_{n+1}] \vdash I\Sigma_m + RFN_{\Sigma_m}(I\Sigma_{m+n-1})$  for  $n > 0$ .

4) The following four principles are equivalent in  $I\Sigma_m$  ( $n > 0$ ):

a.  $RFN_{\Sigma_m}(I\Sigma_{m+n-1})$ ,      b.  $Con(I\Sigma_{m+n-1} + T_m)$ ,

c.  $LSP^*_{\Sigma_m}[\omega_{n+1}]$ ,      d.  $WOP^*_{\Sigma_m}[\omega_{n+1}]$ .

THEOREM 2. 1) For each  $m > 0$ ,  $I\Sigma_m \vdash RFN_{\Sigma_{m+1}}(I\Sigma_{m-1})$ .

2) For each  $k > 0$  and  $m \geq 0$ ,  $I\Sigma_k + T_m \not\vdash RFN_{\Sigma_m}(I\Sigma_k)$  if  $I\Sigma_k + T_m$  is consistent.

To prove Theorem 1. 3), we need to introduce Skolem functions, and reduce the original fragments of arithmetic to fragments in the extended language, having weaker mathematical induction. (As for details, see §3 of [3].)

In the following, we will give a proof of Theorem 1. 2). We assume the familiarity with Ketonen and Solovay [1] and Kurata [2]. We remark that both implications  $c. \Rightarrow b.$  and  $b. \Rightarrow a.$  can be proved in the same way as Theorems 2.5.5 and 2.5.6 in [2]. In either case, we need the  $\Sigma_m$ -mathematical induction  $Ind\Sigma_m$  or the  $\Sigma_m$ -least number principle  $LS_m$ , which is

equivalent to  $\text{Ind}\Sigma_m$ . To show this, we will give here a detailed proof of the implication  $b. \Rightarrow a.$

Let  $T$  be the theory obtained from  $\text{IS}_m$  by adding

$$(1) \quad \forall x ( \forall y ( y <_n x \supset \varphi(y) ) \supset \varphi(x) )$$

and

$$(2) \quad \exists z \neg \varphi(z)$$

as additional axioms, where  $\varphi(z)$  is a  $\pi_m$ -formula. We can suppose that  $\varphi(z)$  is  $\forall u \psi(z, u)$  for a  $\Sigma_{m-1}$ -formula  $\psi(z, u)$ . Then, it follows from (1) that

$$(3) \quad T \vdash \forall x \forall u \exists y \exists v [ \neg \psi(x, u) \supset ( y <_n x \wedge \neg \psi(y, v) ) ] .$$

Let  $J$  be the primitive recursive pairing function defined by  $J(x, y) = \frac{1}{2}[(x+y)^2 + 3x + y]$  and both  $K(z)$  and  $L(z)$  are primitive recursive projection functions satisfying that

$$i. \quad J(K(z), L(z)) = z,$$

$$ii. \quad K(J(x, y)) = x \quad \text{and} \quad L(J(x, y)) = y.$$

Now, define  $\theta(z)$  by  $\neg \psi(K(z), L(z))$ . Clearly,  $\theta(z)$  is a  $\pi_{m-1}$ -formula. From (3) it follows that

$$T \vdash \forall z \exists w ( \theta(z) \supset ( K(w) <_n K(z) \wedge \theta(w) ) ).$$

Let  $\xi(z, w)$  denote the formula

$$\theta(z) \supset ( K(w) <_n K(z) \wedge \theta(w) ).$$

Then,  $\xi(z, w)$  belongs to  $\Delta_m$ . Since

$$\exists w \xi(z, w) \supset \exists w ( \xi(z, w) \wedge \forall u <_w \neg \xi(z, u) )$$

follows from  $\text{LS}_m$ ,

$$T \vdash \forall z \exists ! w ( \xi(z, w) \wedge \forall u <_w \neg \xi(z, u) ).$$

Similarly, since  $T \vdash \exists w \theta(w)$  and moreover  $\exists w \theta(w) \supset \exists w ( \theta(w) \wedge \forall u <_w \neg \theta(u) )$  follows from  $\text{LS}_m$ , we have

$$T \vdash \exists ! w ( \theta(w) \wedge \forall u <_w \neg \theta(u) ).$$

Now define  $\Sigma_m$ -formulas  $\tau(x, t)$  and  $\sigma(x, s)$  by

$$\begin{aligned}\tau(x, t) \equiv & \exists z [ \exists y ( \beta(z, 0) = y \wedge \theta(y) \wedge \forall u < y \neg \theta(u) ) \\ & \wedge \forall u < x \exists v \exists w ( \beta(z, u) = v \wedge \xi(v, w) \wedge \forall r < w \neg \xi(v, r) \\ & \wedge \beta(z, u+1) = w ) \wedge \beta(z, x) = t ],\end{aligned}$$

and

$$\sigma(x, s) \equiv \exists t ( \tau(x, t) \wedge s = K(t) ).$$

( Notice that  $\tau$  and  $\sigma$  represent the graphs of functions  $g$  and  $f$  in the proof of Theorem 2.5.6 in [2], respectively. ) By using  $\text{Ind}\Sigma_m$ , both  $F(\tau)$  and  $F(\sigma)$  are provable in  $T$ . On the other hand,

$$\begin{aligned}T \vdash ( \sigma(x, s) \wedge \sigma(x+1, s') ) \supset & \exists t \exists t' [ s = K(t) \\ & \wedge s' = K(t') \wedge \tau(x, t) \wedge \tau(x+1, t') \wedge \xi(t, t') ].\end{aligned}$$

Clearly,  $\xi(t, t')$  implies  $\theta(t) \supset K(t') <_n K(t)$ , i.e.,  $\theta(t) \supset s' <_n s$ . But by using  $\text{Ind}\Sigma_m$ ,  $T \vdash \tau(x, t) \supset \theta(t)$ . Therefore,

$$T \vdash \forall x \forall s \forall s' ( ( \sigma(x, s) \wedge \sigma(x+1, s') ) \supset s' <_n s ).$$

Hence,  $\text{WOP}_{\Sigma_m} [\omega_n]$  for  $\sigma$  fails in  $T$ . By taking the contraposition, we have

$$I\Sigma_m + \text{WOP}_{\Sigma_m} [\omega_n] \vdash \text{TI}_{\Pi_m} [\omega_n].$$

Next, we will show that  $I\Sigma_m + \text{TI}_{\Pi_m} [\omega_n] \vdash \text{LSP}_{\Sigma_m} [\omega_n]$ . We remark here that  $\text{TI}_{\Pi_m} [\omega_n]$  is equivalent in  $I\Sigma_1$  to the schema

$$(4) \quad \exists x \psi(x) \supset \exists y [ \psi(y) \wedge \forall z ( z <_n y \supset \neg \psi(z) ) ],$$

where  $\psi(x)$  is a  $\Sigma_m$ -formula. Let  $T$  be the theory obtained from  $I\Sigma_m$  by adding the above schema (4) and the formula  $\text{SIF}(\theta)$  for a  $\Sigma_m$ -formula  $\theta$ , as additional axioms. Let  $\alpha$  be an ordinal such that  $\alpha < \omega_n$ . For a given number  $x$ , let  $\theta^*(s, v)$  denote the following  $\Sigma_m$ -formula;

$$\exists z [ \beta(z, 0) = \alpha \wedge \forall w < s \exists u \exists t ( \beta(z, w) = u \wedge \theta(x+w, t)$$

$$\wedge \beta(z, w+1) = fs_n(u, t) ) \wedge \beta(z, s) = v ] .$$

Clearly,  $\theta^*(s, 0)$  means that  $[x, x+s]$  is  $(\alpha, \theta)$ -large. When  $\alpha = 0$ , it is obvious that  $T \vdash \exists s \theta^*(s, 0)$ . So, suppose otherwise.

Let  $\psi(r)$  be  $\exists s \exists v ( \theta^*(s, v) \wedge v <_n r )$ . Then,  $T \vdash \exists r \psi(r)$ .

Thus, by the schema (4)

$$T \vdash \exists r [ \psi(r) \wedge \forall z ( z <_n r \supset \neg \psi(z) ) ] .$$

Take such an  $r$ . Then,  $\exists s \exists v ( \theta^*(s, v) \wedge v <_n r )$ . Take also such  $s$  and  $v$ . Then,  $\neg \psi(v)$  holds. Hence

$$(5) \quad T \vdash \forall s' \forall v' ( \theta^*(s', v') \supset \neg ( v' <_n v ) ) .$$

On the other hand,

$$T \vdash \forall s' \exists w \theta^*(s', w)$$

by using  $\text{Ind}\Sigma_m$ . In particular,  $T \vdash \exists w \theta^*(s+1, w)$ . Thus we have

$$T \vdash \exists s \exists v \exists w ( \theta^*(s, v) \wedge \theta^*(s+1, w) \wedge \neg ( w <_n v ) )$$

by (5). But, fundamental sequences have the property:

$$T \vdash \forall s \forall v \forall w ( ( \theta^*(s, v) \wedge \theta^*(s+1, w) \wedge 0 <_n v ) \supset w <_n v ) .$$

Hence,  $T \vdash \exists s \theta^*(s, 0)$ . Therefore,

$$\forall \alpha < \omega_n \forall x \exists y ( [x, y] \text{ is } (\alpha, \theta)\text{-large} )$$

is provable in  $T$ .

Remark here that  $I\Sigma_1 + \text{TI}_{\Pi_m}[\alpha] \vdash \text{L}\Sigma_m$  if  $\omega \leq \alpha$ . Thus,

$I\Sigma_m + \text{TI}_{\Pi_m}[\omega_n]$  is equivalent to  $I\Sigma_1 + \text{TI}_{\Pi_m}[\omega_n]$  when  $n > 0$ .

Therefore, we have Theorem 1. 2).



## REFERENCES

- [1] J. Ketonen and R. Solovay, Rapidly growing Ramsey functions.  
Annals of Mathematics, vol.113. 1981, pp.267-314.
- [2] R. Kurata, Paris-Harrington principles, reflection  
principles and transfinite induction up to  $\varepsilon_0$ . ( to appear ).
- [3] H. Ono, Reflection principles in fragments of Peano  
arithmetic. ( in preparation ).